Ramanujan congruences for infinite family of partition functions

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Integer partitions

Definition

An (integer) partition of n is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r \geq 1$ that sum to n. Let $p(n)$ be the number of partitions of n. By convention, we take $p(0) = 1$ and $p(n) = 0$ for negative n.

For example, if $n = 4$, $p(4) = 5$.

1. 4
2. 3+1
3. 2+2
4. 2+1+1
5. 1+1+1+1

Motivation

Consider the first 24 values of the partition function $p(n)$

n	**P(n)**	**n**	**P(n)**	**n**	**P(n)**	**n**	**P(n)**	**n**	**P(n)**
0	1	5	7	10	42	15	176	20	627
1	1	6	11	11	56	16	231	21	792
2	2	7	15	12	77	17	297	22	1002
3	3	8	22	13	101	18	385	23	1255
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- Also if you look closely, 7 **divides** $p(5), p(12)$ and $p(19)$.
- 11 **divides** $p(6)$ and $p(17)$.

Introduction

Theorem (Ramanujan 1920s, Watson 1930s, Atkin 1960s)

For all positive integers n, we have,

$$p(5n+4) \equiv 0 \pmod{5},$$

$$p(7n+5) \equiv 0 \pmod{7},$$

$$p(11n+7) \equiv 0 \pmod{11}.$$

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The **generating function** for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \frac{q^{1/24}}{\eta(\tau)}$$

here $q = e^{2\pi i \tau}$. This is a weight $-1/2$ weakly holomorphic modular form on $\Gamma(24)$.

$$\text{Here} \quad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \quad \text{is the Dedekind eta function.}$$

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Theorem (Ono and Ahlgren, 2001)

If $\ell \geq 5$ *is prime,* n *is a positive integer, and* $24\beta \equiv 1 \pmod{24}$*, then there are infinitely many non-nested arithmetic progressions* $\{An + B\} \subset \{\ell n + \beta\}$*, such that for every integer* n *we have*

$$p(An + B) \equiv 0 \pmod{\ell}.$$

Introduction

To study a large class of restricted partition functions, we study the partition function $p_{[1^c \ell^d]}(n)$. This can be defined using generating functions,

$$\sum_{n=0}^{\infty} p_{[1^c \ell^d]}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^c (1-q^{\ell n})^d}.$$

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Examples

- ℓ-Regular partition function $b_\ell(n)$, $c = 1, d = -1$. Ex: $b_3(4) = 4$,

  The generating function $\displaystyle\sum_{n=0}^{\infty} b_\ell(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^{\ell m})}{(1-q^m)}.$

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- ℓ-core partition function $a_\ell(n)$, $c = 1, d = -\ell$. Ex: $a_3(4) := 2$

  The generating function $\displaystyle\sum_{n=0}^{\infty} a_\ell(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^{\ell m})^\ell}{(1-q^m)}.$

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Theorem (Liuquan Wang, 2017)

For any positive integer k and for $n > 0$,

$$b_5\left(5^{2k}m + \frac{5^{2k}-1}{6}\right) \equiv 0 \pmod{5^k}.$$

Theorem (Liuquan Wang, 2016)

$$p_{[1^1 11^{-11}]}(11^k n + 11^k - 5) \equiv 0 \pmod{11^k}$$

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$$p_{[1^1 11^{-1}]}\left(11^{2k-1}n + \frac{7\cdot 11^{2k-1}-5}{12}\right) \equiv 0 \pmod{11^k}$$

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Furthermore, Wang stated that it should be possible to obtain congruences for the partition function $p_{[1^c 11^d]}(n)$. However Wang proved each case separately.

Main Result

Our goal was to derive a proof that works for all the cases and obtain a similar result for the other primes less than or equal to 13.

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Theorem

For $\ell \leq 13$ a prime, for any positive integer r and for integers c, d such that $c > 0$ and $d \geq -2$,

$$p_{[1^c \ell^d]}(\ell^r m + n_r^\ell) \equiv 0 \pmod{\ell^{A_r^\ell}}$$

where $24n_r^\ell \equiv (c + \ell d) \pmod{\ell^r}$. For $\ell = 11$ this is true for all integers c, d.

Here A_r^ℓ depends on the prime ℓ, the integers c, d and can be calculated explicitly.

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Here I only talk about the case $\ell = 11$ in detail and at the end I will briefly talk about the case $\ell = 5$.

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In 1981, Basil Gordon proved congruences for the partition function $p_{-k}(n)$. The generating function for the partition function $p_{-k}(n)$ is given by,

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Theorem (Gordon 1981)

If $24n \equiv k \pmod{11^r}$,

$$p_{-k}(n) \equiv 0 \pmod{11^{\frac{\alpha r}{2}+\epsilon}}$$

where $\epsilon = \epsilon(k) = O\big(\log|k|\big)$, *if* $k \geq 0, \alpha$ *depends on the residue of* $k \pmod{120}$ *according to the following table.*

Preliminaries

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	2	1	2	1	1	1	2	2	1	1	2	2	1	2	1	0	0	1	1	0	0	1	1	0
24	1	1	1	1	2	2	1	1	2	2	1	0	0	0	0	1	1	0	0	1	1	1	0	0
48	1	1	2	2	1	1	1	0	1	0	1	0	0	1	1	0	0	1	0	1	0	1	0	0
72	2	1	1	1	2	1	2	1	2	1	2	2	1	1	1	2	1	2	1	2	1	1	1	0
96	0	0	1	0	1	0	1	0	1	1	0	0	0	1	0	1	0	1	0	1	1	0	0	0

Table: 1

Here the entry is $\alpha(24i + j)$ where row labelled $24i$ and column labeled j. When $k < 0$, the last column must be changed to $2, 2, 2, 0, 2$.

Preliminaries

The U_p Operator

For a Laurent series $f(\tau) = \sum_{n \geq N} a(n)q^n$, we define the U_p operator by,

$$U_p\left(f(\tau)\right) = \sum_{pn \geq N} a(pn)q^n.$$

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Let $g(\tau) = \sum_{n \geq N} b(n)q^n$ be an another Laurent series.

$$U_p\left(f(\tau)g(p\tau)\right) = g(\tau)U_p\left(f(\tau)\right).$$

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Theorem (Atkin-Lehner)

If $f(\tau)$ is a modular function for $\Gamma_0(N)$, if $p^2|N$, then $U_p\Big(f(\tau)\Big)$ is a modular function for $\Gamma_0(N/p)$.

Preliminaries

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Lemma (Gordon,1981)

For all $v \in \mathbb{Z}$

1. $J_\nu(\tau) = J_{\nu-5}(\tau) J_5(\tau)$,
2. $\{J_\nu(\tau) | -\infty < \nu < \infty\}$ *is a basis for* V
3. $Ord_\infty J_\nu(\tau) = \nu$
4. $$ord_0 J_v(\tau) = \begin{cases} -\nu & \text{if } \nu \equiv 0 \pmod 5 \\ -\nu - 1 & \text{if } \nu \equiv 1, 2 \text{ or } 3 \pmod 5 \\ -\nu - 2 & \text{if } \nu \equiv 4 \pmod 5 \end{cases}$$
5. *The Fourier series of* $J_\nu(\tau)$ *has integer coeffients, and is of the form* $J_\nu(\tau) = q^\nu + \ldots$

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V is mapped to itself by the linear transfomation,

$$T_\lambda : f(\tau) \rightarrow U_{11}\left(\phi_{11}(\tau)^\lambda f(\tau)\right)$$

here λ is an integer and $\phi_{11}(\tau) = \frac{\eta(121\tau)}{\eta(\tau)}$.

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Let $(C^\lambda_{\mu,\nu})$ be the matrix of the linear transfomation T_λ with respect to the basis elements J_ν.

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Gordon obtained these recurrences for the matrix elements,

$$C^{\lambda+12}_{\mu-5,\nu+5} = C^\lambda_{\mu,\nu}$$

$$C^\lambda_{\mu,\nu} \equiv C^{\lambda-11}_{\mu,\nu-5} \pmod{11}.$$

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here $\delta = \delta(\mu, \nu)$ depends on the residues of μ and ν (mod 5) according to table 2.

	ν				
μ	**0**	**1**	**2**	**3**	**4**
0	-1	8	7	6	15
1	0	9	8	2	11
2	1	10	4	3	12
3	2	6	5	4	13
4	3	7	6	5	9

Table: 2

Preliminaries

Now by the Lemma , the Fourier series of $T_\lambda(J_\mu)$ has all coefficients divisible by 11 if and only if,

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Now we define;

$$\theta(\lambda,\mu) = \begin{cases} 1 & \text{if all the coefficients of } U_{11}(\phi^\lambda J_\mu) \text{ divisible by 11} \\ 0 & \text{otherwise} \end{cases}$$

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2	1	1	1	0	0	0	0	1	1	0	0
3	1	0	1	0	0	0	0	1	1	0	0
4	1	0	1	0	1	0	1	1	0	0	0

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Key Ideas

Our first goal is to construct sequence of modular functions that are the generating functions for the partitions $p_{[1^c 11^d]}(n)$.

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$$L_2(\tau) = \prod_{n=1}^{\infty}(1-q^{11n})^d(1-q^n)^c \sum_{m\geq\mu_2}^{\infty} p_{[1^c 11^d]}(11^2m+n_2)q^m$$

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for any positive integer r and integers c, d. We also put $A_0 = 0$.

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$$\text{We can prove} \quad \pi(L_r) \geq A_r.$$

Key Ideas

By the recurrence relation between L_{2r} and L_{2r-1},

$$n_{2r} = -5d \cdot 11^{2r-1} + n_{2r-1}$$

$$n_{2r-1} = -5c \cdot 11^{2r-2} + n_{2r-2}$$

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From this we have that,

$24n_{2r-1} \equiv (c+11d) \mod 11^{2r-1}$ and $24n_{2r} \equiv (c+11d) \mod 11^{2r}$

Therefore n_r are integers such that,

$$24n_r \equiv (c+11d) \pmod{11^r}.$$

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Example ($c = 1, d = -11$)

In this case, λ_i is 1 if i even or is -11 if i is odd.
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Example ($c = 2, d = 7$)

$$p_{[1^2 11^7]}\left(11^{2r}m - \frac{7 \cdot 11^{2r} - 79}{24}\right) \equiv 0 \pmod{11^{2r-1}}.$$

Congruences for $\ell = 5$

In this case we define $\theta(b) = \begin{cases} 1 & \text{if } b \equiv 1 \text{ or } 2 \pmod 5, \\ 0 & \text{Otherwise.} \end{cases}$

We also define, for $r \geq 1$,

$$A_{2r-1} = \theta(c) + \sum_{i=1}^{r-1} \{\theta(6k_i + 6 + d) + \theta(6l_i + 6 + c)\}, \quad A_{2r} = A_{2r-1} + \theta(6k_i + 6 + d),$$

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Example

For 5-regular partitions $b_5\left(5^{2r}m + \frac{5^{2r}-1}{6}\right) \equiv 0 \pmod{5^r}$

For 5-core partitions $a_5(5^r m - 1) \equiv 0 \pmod{5^r}$

Questions/Future Work

There are two ways to prove the congruences for $p_{[1^c \ell^d]}(n)$ for the other primes,

- Construct bases for modular functions on $\Gamma_0(\ell)$ and use the Gordon's method to prove the congruences.
- Use modular forms modulo ℓ theory.

Theorem (Folsom, Kent, Ono, 2012)

$$\textit{Let} \quad L_0 := 1 \quad \textit{and} \quad L_r := U_\ell \left(\phi_\ell^{\lambda_{r-1}}(\tau) L_{r-1} \right)$$

$$\textit{here} \quad \phi_\ell(\tau) := \frac{\eta(\ell^2 \tau)}{\eta(\tau)} \quad \textit{and} \quad \lambda_r = \begin{cases} 1 & \textit{if } r \textit{ is even}, \\ 0 & \textit{if } r \textit{ is odd.} \end{cases}$$

If $m \geq 1$, $5 \leq \ell \leq 31$ and $r \geq m^2$, then L_r belongs to a $\mathbb{Z}/\ell^m\mathbb{Z}$-module with rank $\leq \left\lfloor \frac{\ell-1}{12} \right\rfloor$.

THANK YOU!